On the Casimir energy of a massive scalar field in positive curvature space

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Abstract

We re-evaluate the zero point Casimir energy for the case of a massive scalar field in $\mathbf{R}^1 \times \mathbf{S}^3$ space, allowing also for deviations from the standard conformal value $\xi = 1/6$, by means of zero temperature zeta function techniques. We show that for the problem at hand this approach is equivalent to the high temperature regularisation of the vacuum energy.

Recently the vacuum and finite temperature energies for massless and massive scalar fields in positive curvature spaces, as $\mathbf{S}^3$, were considered, with special emphasis being put on the analysis of entropy bounds corresponding to those cases [see Refs. [1] and [2], respectively]. Specifically, in Ref. [2], dealing with the massive case and arbitrary coupling, the vacuum (Casimir) energy and the finite temperature energy were obtained through the application of a generalised zeta function technique [3], which provides a neat formal separation of the logarithm of the partition function into the non-thermal and thermal sectors. This approach holds generally and, in particular, it does for $\mathbf{S}^1 \times \mathbf{S}^d$ spaces. For the special case $d = 3$, the result for the logarithm of the partition function can be combined with the Abel-Plana rescaled sum formula [5], leading to [2]

$$\begin{aligned} \log Z(\beta) &= -\frac{\beta}{2r}\sum_{n=1}^{\infty} n^2\left(n^2+\mu_{eff}^2\right)^{1/2} + \frac{r\mu_{eff}^2}{\beta}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2\left(m_{eff}\beta n\right) \\ &+ \frac{\mu_{eff}^2\beta}{(2\pi)^2 r}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2\left(2\pi n\mu_{eff}\right) - \frac{\mu_{eff}^3\beta}{2\pi r}\sum_{n=1}^{\infty}\frac{1}{n}K_3\left(2\pi n\mu_{eff}\right). \end{aligned} \tag{1}$$

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From this result the present authors conjectured that the *renormalised* vacuum energy could be inferred as

$$E_0 = -\frac{\mu_{eff}^2}{(2\pi)^2 r}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2\left(2\pi n\mu_{eff}\right) + \frac{\mu_{eff}^3}{2\pi r}\sum_{n=1}^{\infty}\frac{1}{n}K_3\left(2\pi n\mu_{eff}\right), \tag{2}$$

where r is the radius of $\mathbf{S}^3$, β is the reciprocal of the temperature, and the parameter μ_{eff}, to be defined below, plays the role of an 'effective mass'. The conjecture stems from the fact that in Eq. (1), the terms linear in β give rise to temperature independent terms when we calculate basic thermodynamics quantities such as the free energy and the energy. Also, in the very-high temperature limit, it is physically plausible to expect the Stefan-Boltzmann term —contained in the second term in Eq. (1)— to be the only surviving one.

here we show that the above-mentioned conjecture holds true and that the standard zero-temperature zeta function regularisation procedure (as prescribed in [3], for instance) works well, yielding the correct vacuum energy for the non-conformal case.

The vacuum energy for a massive scalar field in $\mathbf{S}^3$ and arbitrary conformal parameter ξ is given by

$$E_0 = \frac{1}{2r}\sum_{\ell=0}^{\infty} D_\ell M_\ell, \tag{3}$$

where

$$M_\ell^2 := (\ell+1)^2 + \mu_{eff}^2 \tag{4}$$

and the dimensionless parameter μ_{eff}^2 is defined as

$$\mu_{eff}^2 = \mu^2 + \chi - 1, \tag{5}$$

where $\mu := mr$, m being the mass of an elementary excitation of the scalar field, r the radius of $\mathbf{S}^3$, and $\chi := \xi\mathcal{R}r^2$; $\mathcal{R}$ is the Ricci curvature scalar. Notice that μ_{eff}^2 and χ are real and $\mu^2 \geq 0$. The degeneracy factor is $D_\ell = (\ell+1)^2$. Hence, for a massive scalar field the unregularised Casimir energy is given by

$$E_0\left(\mu_{eff}^2\right) = \frac{1}{2r}\sum_{\ell=0}^{\infty}(\ell+1)^2\sqrt{(\ell+1)^2 + \mu_{eff}^2}. \tag{6}$$

The standard conformal case corresponds to the values $\mu^2 = 0$ and $\chi = 1$ ($\xi = 1/6$, $\mathcal{R} = 6/r^2$) and, using the zeta function prescription, we obtain

$$E_0\left(\mu^2 = 0, \chi = 1\right) = \frac{1}{2r}\zeta(-3) = \frac{1}{240r}. \tag{7}$$

The problem now is to find an analytical continuation in terms of the zeta function for the more difficult series in Eq. (6).

Starting from Eq. (6) we can straightforwardly write:

$$\sum_{\ell=0}^{\infty} (\ell+1)^2 \left[(\ell+1)^2 + \mu_{eff}^2\right]^{-s}\Big|_{s=-\frac{1}{2}} = F(-3/2, \mu_{eff}^2) - \mu_{eff}^2 F(-1/2, \mu_{eff}^2), \tag{8}$$

where by definition,

$$F(s, \mu_{eff}^2) := \sum_{\ell=0}^{\infty} \left[(\ell+1)^2 + \mu_{eff}^2\right]^{-s} \tag{9}$$

The sum over ℓ can be identified as an Epstein series whose analytical continuation is well-known [3, 4]. It follows that the vacuum energy is

$$E_0 = -\frac{\mu_{eff}^4}{16r}\Gamma(-2) + \frac{3\mu_{eff}^2}{4\pi^2 r}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2(2\pi n\mu_{eff}) + \frac{\mu_{eff}^3}{2\pi r}\sum_{n=1}^{\infty}\frac{1}{n}K_1(2\pi n\mu_{eff}). \tag{10}$$

The divergent term in Eqs. (10) can be taken care of with the minimal subtraction scheme [9]. However, this procedure would mean, in our case, to keep a quartic term in μ_{eff} that would spoil the behaviour of the vacuum energy in the classic limit where the vacuum oscillations must vanish. Therefore, without too much ado we will simply discard this term. Now, we can further simplify Eq. (10) and write

$$\begin{aligned} E_0 &= \frac{\mu_{eff}^2}{4\pi^2 r}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2(2\pi n\mu_{eff}) \\ &+ \frac{\mu_{eff}^3}{4\pi r}\sum_{n=1}^{\infty}\frac{1}{n}K_1(2\pi n\mu_{eff}) + \frac{\mu_{eff}^3}{4\pi r}\sum_{n=1}^{\infty}\frac{1}{n}K_3(2\pi n\mu_{eff}). \end{aligned} \tag{11}$$

If we now make use of the recursion relation

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z}K_\nu(z),$$

we easily obtain Eq. (2), as conjectured in a former paper [2].

If we take the limit $\mu_{eff}^2 \to 0$, Eqs. (2) and (11) yield the well known result $E_0 \approx 1/240$. If we make use of the appropriate small argument expansion of the Bessel functions of the second kind we obtain

$$E_0 \approx \frac{1}{240r} - \frac{\mu_{eff}^2}{48r} - \frac{1}{2}\left[\frac{1}{8r} + \frac{1}{16r}\left(-\frac{3}{2} + 2\gamma_E\right)\right]\mu_{eff}^4. \tag{12}$$

If we take the opposite limit, $\mu_{eff}^2 \to \infty$, the vacuum energy given by Eq. (2) —or (10) or (11)— behaves in the way that one would normally expect of constrained zero-point oscillations of a massive quantum field: the vacuum energy goes to zero in an exponential way.

A detailed version of the calculation briefly described here can be found in [10].

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